

Non-perturbative scaling in the scalar theory

Alfio Bonanno

Osservatorio Astrofisico, Università di Catania

Viale Andrea Doria 6, 95125 Catania, Italy

INFN, sezione di Catania

Corso Italia 57, 95128 Catania, Italy

Abstract

A new approach to study the scaling behavior of the scalar theory near the Gaussian fixed point in d -dimensions is presented. For a class of initial data an explicit use of the Green's function of the evolution equation is made. It is thus discussed under which conditions non-polynomial relevant interactions can be generated by the renormalization group flow.

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Recent studies have discussed the possibility that at the Gaussian Fixed Point (GFP) of the $O(N)$ invariant scalar theory new relevant eigen-directions exist. In particular in [1] Halpern and Huang have found a class of non-polynomial relevant interactions at the GFP by means of Wilson's renormalization group (RG) transformation with the "sharp" momentum shell integration, but their result is still much debated [2,3].

In fact an interesting question to study is the general features of the scaling around the GFP. In particular it would be important to have a global understanding of all the possible solutions in order to discuss the structure of the continuum limit.

A necessary condition to describe this limit is to select a set of marginal and relevant scaling eigen-operators at this fixed point. A scaling eigen-operator is a solution of the linearised fixed point equation of the form

$$u = \left(\frac{\Lambda}{k}\right)^\nu h(x) \quad (1)$$

where x is the dimensionless field, Λ is the UV overall cut-off, k is the running cut-off and ν is a scaling exponent. While in perturbation theory only a finite number of polynomial relevant and marginal interactions are considered, it is still not clear if new non-perturbative non-polynomial relevant scaling interactions can be generated by the RG flow. Were this true, it would mean for instance that the polynomial interactions cannot span all the continuum physics.

The aim of this work is to clarify some aspects of this question. Instead of using the methods discussed in [1,5,6], the Green's function of the linearised renormalization group equation is constructed. It will be thus shown the role played by the boundary conditions in determining the structure of the scaling fields and it will be shown that is possible to generate relevant non-polynomial interactions not belonging to the same universality class of the polynomial interactions.

The most convenient way to address such a question is to use the Wilsonian formulation of the RG transformation. In particular we consider the local potential approximation (LPA) of Wegner-Houghton [4] equation as discussed in [5] for the $N = 1$ components scalar theory, but we shall not restrict the potential to a polynomial.

The flow equation for the scaling field $u(x, t)$ reads [5,6]

$$\frac{\partial u}{\partial t} = d u - \frac{d-2}{2} x \frac{\partial u}{\partial x} + a_d \frac{\partial^2 u}{\partial x^2} \quad (2)$$

where $d > 2$ is the dimension of the spacetime, Λ is the UV cut-off, $a_d = 1/(2\sqrt{\pi})^d \Gamma(d/2)$, and $t \equiv \ln \Lambda/k$. The GFP is at $u \equiv 0$. In studying the UV region near the GFP one is interested in finding the unique function $u(x, t)$ which is continuous in the closed upper half plane $-\infty < x < \infty$, $0 \leq t$ and satisfies eq.(2) with initial condition $u(x, 0) = f(x)$, $-\infty < x < \infty$ with $f(x) \in C^0$. We shall leave open the possibility that f is not bounded since there are no physical reasons to assume $|u(\pm\infty, t)| < \infty$ in discussing the scaling interactions.

It is convenient to make the transformation

$$u(x, t) = v(x, t) e^{(3d-2)t/2} \quad (3)$$

which brings (2), in the equivalent Fokker-Planck form [7]

$$\frac{\partial v}{\partial t} = -\frac{d-2}{2} (v + x \frac{\partial v}{\partial x}) + a_d v_{xx} \quad (4)$$

After the rescaling $x \rightarrow x \sqrt{(d-2)/2a_d}$ and $t \rightarrow t (d-2)/2 \equiv \tau$, it reads

$$\frac{\partial v}{\partial \tau} = L_{\text{FP}}[v], \quad L_{\text{FP}} \equiv \frac{\partial}{\partial x} \left[\frac{\partial}{\partial x} - x \right] \quad (5)$$

where we have introduced the Fokker-Planck operator L_{FP} . It is not difficult to build a Green's function for such an operator. Let us define the formally self-adjoint operator

$$L = e^{-x^2/4} L_{\text{FP}} e^{x^2/4} = - \left[-\frac{\partial^2}{\partial x^2} + \frac{x^2}{4} + \frac{1}{2} \right] \equiv -H_{\text{ho}} - \frac{1}{2} \quad (6)$$

being H_{ho} the Hamiltonian of the quantum mechanical harmonic oscillator with $\hbar = \omega = 1$ and mass $m = 1/2$. We shall refer to the description in terms of such an operator and eigenfunctions as the quantum-mechanical (QM) frame, as opposed to the Fokker-Planck (FP) frame in eq.(5). The eigenfunctions of L have the asymptotic behavior $\sim e^{\pm x^2/4}$ as $|x| \rightarrow \infty$. For L to be self-adjoint only the damped exponential must be chosen, and the eigenfunctions are known from Quantum Mechanics. In particular the spectrum of L satisfies $L \varphi_n = -(n+1)\varphi_n$ with $n = 0, 1, 2, \dots$ and all the eigenvalues of L are eigenvalues of L_{FP} with eigenfunctions $L_{\text{FP}}\phi_n = -(n+1)\phi_n = e^{x^2/4}\varphi_n$. Therefore a complete and orthonormal set of eigenfunctions for the L_{FP} operator reads

$$\phi_n(x) = \frac{1}{4\sqrt{2\pi} \sqrt{2^n n!}} H_n(x/\sqrt{2}) \quad (7)$$

where H_n are the Hermite polynomials, with the orthonormality condition

$$\langle \phi_n | \phi_m \rangle = \int_{-\infty}^{+\infty} e^{-x^2/2} \phi_n \phi_m dx = \delta_{mn} \quad (8)$$

Thus the linear space generated by the set ϕ_n is $\mathcal{L}^2(\mathcal{R})$ with the norm $\|f\| = \sqrt{\langle f | f \rangle}$. It is now straightforward to write down the Green's function $G(x, \tau | y, 0)$ of the Fokker-Planck operator from the well known expression of the Green's function of the harmonic oscillator. We have

$$G(x, \tau | y, 0) = \langle x | e^{\tau L_{\text{FP}}} | y \rangle = e^{-y^2/2} \sum_n e^{-\tau(n+1)} \phi_n(x) \phi_n(y) \quad (9)$$

$$= e^{-y^2/2} e^{x^2/4} \sum_n e^{-\tau(n+1)} \varphi_n(x) \varphi_n(y) e^{y^2/4} \quad (10)$$

$$= e^{x^2/4} \langle x | e^{-\tau(H_{\text{ho}} + \frac{1}{2})} | y \rangle e^{-y^2/4 - \tau/2} \quad (11)$$

$$= \sqrt{\frac{e^{-\tau}}{4\pi \sinh \tau}} \exp \left(-\frac{e^{\tau}}{4 \sinh \tau} (x e^{-\tau} - y)^2 \right) \quad (12)$$

This is precisely what we find in [7] with a summation formula of the Hermite polynomials. Therefore the unique solution of (5) which is continuous and satisfies the following initial conditions $v(x, 0) = f(x)$ with $|\langle \phi_n | f \rangle| < \infty$ is given by

$$v(x, \tau) = \int_{-\infty}^{+\infty} G(x, \tau; y, 0) f(y) dy \quad (13)$$

We now come to the main questions of our investigation, namely to study the subspace of relevant and marginal interactions present at the fixed point. It is not difficult to see that *the solutions of eq.(2) that $\in \mathcal{L}^2(\mathcal{R})$ which are not bounded as $t \rightarrow +\infty$ have $n \leq 2d/(d-2)$ in the spectrum of L_{FP} .*

The proof is a consequence of the spectral properties of the flow evolution operator: given $f \in \mathcal{L}^2(\mathcal{R})$ as an initial condition, from (3) and (13) one has

$$u(x, t) = e^{(3d-2)t/2} \int_{-\infty}^{+\infty} G(x, \tau|y, 0) f(y) dy \quad (14)$$

$$= e^{(3d-2)t/2} \int_{-\infty}^{+\infty} \sum_n e^{-\tau(n+1)} \phi_n(x) \phi_n(y) f(y) e^{-\frac{y^2}{2}} dy \quad (15)$$

$$= \sum_n e^{(d-(d-2)n/2)t} \phi_n(x) \langle \phi_n | f \rangle \quad (16)$$

where we have used eq.(11) and the uniform convergence of the sum. From the last line we see that in order to have a solution that grows for $t > 0$ it must be $n \leq 2d/(d-2)$. If we set $f = \phi_m$ we recover the subspace of marginal and relevant eigenvectors spanned by the first $n \leq 2d/(d-2)$ Hermite polynomials, as it has already found in the previous literature [5,6]. Non-polynomial interactions may be evolved by means of the Green's function provided $|\langle \phi_n | f \rangle| < \infty$, but these will not be of the form (1) and therefore cannot be considered as scaling fields. More specifically if we try to evolve an initial datum that behaves like $\exp(qx^2)$ for $x \rightarrow \infty$ with $q < 1/2$, we find $u(x, t) \sim e^{(3d-2)t/2} \exp(e^{-(d-2)t} qx^2)$ as $t \rightarrow \infty$ (note that $|\langle \phi_n | f \rangle| = \infty$ if $q \geq 1/2$).

We want now to show that the space where all the possible physically meaningful scaling interactions live is greater than $\mathcal{L}^2(\mathcal{R})$. The key point to be noticed is that *the QM frame does not reproduce all the possible solutions present in the FP frame*. This is because L is self-adjoint for the boundary conditions at infinity of the type $\varphi_n \rightarrow e^{-x^2/4}$. In this case from Sturm-Liouville theory we know that the spectrum is bounded and the number of relevant and marginal interactions is thus finite [8]. However L_{FP} is not self-adjoint and its spectrum is greater than the self-adjoint extension of L . In fact, although there are no non-stationary solutions of the linearised equation in the QM frame, a non-trivial zero mode is present in the spectrum of the L_{FP} operator, namely

$$v_0 = e^{x^2/2} \quad (17)$$

which of course $\notin \mathcal{L}^2(\mathcal{R})$ and it corresponds to the following non-stationary solution of the original equation (2)

$$u(x, t) = e^{(3d-2)t/2} e^{(d-2)x^2/2a_d} \quad (18)$$

where we have inserted the factor $\sqrt{(d-2)/2a_d}$ in the definition of x . This is a perfectly well defined and regular scaling field which is “relevant” because it grows in the IR, with scaling dimension $\nu = (3d-2)/2$, and it does not belong to the subspace spanned by the ϕ_n eigenfunctions previously constructed. It is therefore a non-perturbative scaling field

(the connection of this solutions with the type discussed in [1] is not immediately clear to me although their asymptotic behavior for large x is similar to what discussed in [1]). In Quantum Mechanics or in Ornstein-Uhlenbeck diffusion processes one would discard such a solution because of the boundary conditions at infinity. In our context there is no specific reason for not considering scaling fields of this type.

Given this particular zero mode of the L_{FP} operator it is immediate to write down the general solution of the homogeneous equation $L_{\text{FP}}[v] = 0$ which is of the form

$$v(x, \tau) = e^{x^2/2} \left(A + B \operatorname{erf} \left(\frac{x}{\sqrt{2}} \right) \right) \quad (19)$$

being A and B two arbitrary constants of integration. More generally if we set $v_n(x, \tau) = e^{\lambda\tau} h_\lambda(x)$ eq.(5) reads

$$h''(x) - xh'(x) - (\lambda + 1)h(x) = 0 \quad (20)$$

For integer and non-negative values of λ a particular solution of the homogeneous equation (20) can always be found with the Laplace method. One explicitly finds for $\lambda = 1, 2, 3, 4$

$$\begin{aligned} h_1 &= e^{x^2/2} x & h_2 &= e^{x^2/2} (1 + x^2) \\ h_3 &= e^{x^2/2} (3x + x^3) & h_4 &= e^{x^2/2} (3 + 6x^2 + x^4) \end{aligned} \quad (21)$$

It should be stressed that solutions of the type (21) which are even function of x are bounded from below and a linear combination may develop non-trivial minima at some value of the field. This is the case of $h_4 e^{4\tau} + c h_2 e^{2\tau}$ for some negative values of the constant c . The general solution is

$$v_n(x, \tau) = e^{n\tau} \left(A h_n(x) + B h_n(x) \int^x e^{s^2/2} h_n(s)^{-2} ds \right) \quad (22)$$

where h_n is obtained with the Laplace method and $n = 0, 1, 2, \dots$.

The presence of solutions of the flow equation with entirely different physical implications should not come as a surprise. In $d = 2$ eq.(2) admits the following solutions, as it can be checked by direct substitution,

$$u(x, t) = e^{(2-\beta^2/4\pi)t} \cos(\beta x) \quad (23)$$

$$u(x, t) = e^{(2+\gamma/4\pi)t} e^{\sqrt{\gamma}x} \quad (24)$$

where β and γ are arbitrary constants. The first one describes the physics of the Sine-Gordon theory, while the second one defines the Liouville theory [9]. Also in this case the same fixed point may describe entirely different physical theories.

The physical relevance (if any) of these solutions is not clear to us. Although one may speculate about their possible applications to the Standard Model or to Cosmology, all we can say for the moment is that the structure of the GFP for a simple scalar theory is richer than previously discussed. In particular, the universality class spanned by the polynomial eigen-potentials in the LPA does not contain all the possible relevant interactions.

One final remark is the following. We have used the linearised Wegner-Houghton equation in the LPA approximation with sharp cut-off. This equation is the same of the Polchinski equation obtained with the smooth cut-off, apart for the unimportant constant a_d [3]. Thus our result is quite robust since these new scaling exponents does not depend on the cut-off function used in the derivation.

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